JOURNAL OF APPROXIMATION THEORY 59, 300–306 (1989)

Some Properties of Two-Dimensional Bernstein Polynomials

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Communicated by P. L. Butzer

Received June 18, 1987; revised March 9, 1988

Let $B_{n,m}f$ be the Bernstein polynomial of two variables, of degree (n, m), corresponding to a function $f: S \to R$, where S is the unit square $[0, 1] \times [0, 1]$. We demonstrate that, if all the partial derivatives of f of the first r orders exist and are continuous in S, then certain polynomials with integral coefficients, $B_{n,m}^{(q, r-q), e}f$, converge to $(\partial^r/\partial x^q\, \partial y^{r-q})f$ both in the uniform and L_p-norms. We define $B_{n,m}^{(q, r-q), e}f$ to be the polynomial $(\partial^r/\partial x^q\, \partial y^{r-q})(B_{n,m}f)$ with each coefficient replaced by its integral part. An estimate of the order of approximation is also obtained.

1. INTRODUCTION AND NOTATION

Let the function of two real variables f be given over the unit square

$$S: [0, 1] \times [0, 1];$$

then the Bernstein polynomial of two variables, of degree (n, m), corresponding to the function f, is defined by means of the formula

$$(B_{n,m}f)(x, y) = \sum_{i=0}^{n} \sum_{k=0}^{m} f\left(\frac{i}{n}, \frac{k}{m}\right) p_{i,n}(x)\, p_{k,m}(y),$$

where

$$p_{j,s}(z) = \binom{s}{j} z^j (1-z)^{s-j}.$$

Obviously

$$\sum_{j=0}^{s} p_{j,s}(z) = 1.$$

0021-9045/89 $3.00

In [4] Butzer has given a more direct proof of a theorem of Kingsley [7], which states that, if all the partial derivatives of f of order $\leqslant r$ exist and are continuous in S, then

$$(\partial^r/\partial x^q\, \partial y^{r-q})(B_{n,m} f) \to (\partial^r/\partial x^q\, \partial y^{r-q}) f \equiv f^{(r)}_{q,r-q}$$

uniformly in S, as n, m approach infinity in any manner whatsoever.

The main purpose of this paper is to show the convergence of the polynomials with integral coefficients $B^{(q,r-q),e}_{n,m} f$, to $f^{(r)}_{q,r-q}$, both in the uniform and L_p-norms.

With Kantorovic, we define $B^{(q,r-q),e}_{n,m} f$ to be the polynomial

$$\begin{aligned}(B^{(q,r-q)}_{n,m} f)(x, y) &\equiv [(\partial^r/\partial x^q\, \partial y^{r-q})(B_{n,m} f)](x, y)\\ &= n(n-1)\cdots(n-q+1)\, m(m-1)\cdots(m-r+q+1)\\ &\quad \cdot \sum_{i=0}^{n-q} \sum_{k=0}^{m-r+q} \mathop{\triangle}_{(x,n^{-1})}^{q} \mathop{\triangle}_{(y,m^{-1})}^{r-q} f\left(\frac{i}{n}, \frac{k}{m}\right)\\ &\quad \cdot p_{i,n-q}(x)\, p_{k,m-r+q}(y),\end{aligned}$$

with each coefficient replaced by its integral part (see [4, 6]). Throughout the following, the superscript "e" will denote a polynomial with integral coefficients in the above sense; and $\|\cdot\|_p$ will denote the $L_p[S]$ norm $(1 \leqslant p \leqslant \infty)$.

2. Preliminary Results

LEMMA 1. *If f is continuous in S and $B_{n,m} f$ is the Bernstein polynomial of f, then*

$$\begin{aligned}|(B_{n,m} f)(x, y) - f(x, y)| &\leqslant \tfrac{3}{2}[\omega^{(1)}(f; n^{-1/2}) + \omega^{(2)}(f; m^{-1/2})]\\ &\leqslant 3\omega(f; n^{-1/2}, m^{-1/2}),\end{aligned}$$

where $\omega^{(i)}(f; \delta)$ are the partial moduli of continuity of f, i.e.,

$$\omega^{(1)}(f; \delta) = \omega(f; \delta, 0) = \sup_{y} \sup_{|x_1 - x_2| \leqslant \delta} |f(x_1, y) - f(x_2, y)|$$

$$\omega^{(2)}(f; \delta) = \omega(f; 0, \delta) = \sup_{x} \sup_{|y_1 - y_2| \leqslant \delta} |f(x, y_1) - f(x, y_2)|$$

and $\omega(f; \delta, \varepsilon)$ is the complete modulus of continuity of f.

The proof follows simply by extending the proof of the one-dimensional Popoviciu theorem [9] to two dimensions and bearing in mind that the complete modulus of continuity $\omega(f; \delta, \varepsilon)$ of the function f is connected with its partial moduli of continuity $\omega(f; \delta, 0)$ and $\omega(f; 0, \varepsilon)$ by the inequalities

$$\omega(f; \delta, \varepsilon) \leqslant \omega(f; \delta, 0) + \omega(f; 0, \varepsilon) \leqslant 2\omega(f; \delta, \varepsilon).$$

We now define the polynomial

$$\bar{B}_{n,m}^{(q,r-q)} f = \frac{(n-q)!}{n!} \frac{(m-r+q)!}{m!} n^q m^{r-q} B_{n,m}^{(q,r-q)} f$$

and let $\bar{B}_{n,m}^{(q,r-q),e} f$ be its corresponding polynomial with integral coefficients.

Lemma 2. *Let f be a function such that its partial derivatives of the first r orders exist and are continuous in S.*

(a) *If $1 \leqslant p < \infty$, then*

$$\|f_{q,r-q}^{(r)} - \bar{B}_{n,m}^{(q,r-q),e} f\|_p \leqslant A + B + O((nm)^{-1/p}), \tag{1}$$

where $A = 3\omega(f_{q,r-q}^{(r)};\ (n-q)^{-1/2},\ (m-r+q)^{-1/2})$, $B = \omega(f_{q,r-q}^{(r)};\ q/n, (r-q)/m)$;

(b) *If $p = \infty$ and the numbers $n^q m^{r-q} \triangle_{(x, n^{-1})}^{q} \triangle_{(y, m^{-1})}^{r-q} f(u_1, u_2)$ are integers, where u_1 is either zero or $(n-q)/n$ and u_2 is either zero or $(m-r+q)/m$, then*

$$\|f_{q,r-q}^{(r)} - \bar{B}_{n,m}^{(q,r-q),e} f\|_\infty \leqslant A + B + O((nm)^{-1}), \tag{2}$$

where A and B are as in (a).

Proof. By ((14) Lemma 2, [4]), we find that

$$\begin{aligned}(\bar{B}_{n,m}^{(q,r-q)} f)(x, y) &= (B_{n-q, m-r+q} f_{q,r-q}^{(r)})(x, y) \\ &\quad + \sum_{i=0}^{n-q} \sum_{k=0}^{m-r+q} \left[f_{q,r-q}^{(r)}(x_{ik}, y_{ik}) - f_{q,r-q}^{(r)}\left(\frac{i}{n-q}, \frac{k}{m-r+q}\right)\right] \\ &\quad \times p_{i,n-q}(x)\, p_{k,m-r+q}(y),\end{aligned}$$

for some x_{ik} and y_{ik} satisfying

$$i/n < x_{ik} < (i+q)/n \qquad \text{and} \qquad k/m < y_{ik} < (k+r-q)/m.$$

In this expression, the first term represents the Bernstein polynomial in two variables corresponding to the function $f_{q,r-q}^{(r)}$. If we denote by

$\omega(f^{(r)}_{q,r-q}; \delta, \varepsilon)$ the complete modulus of continuity of $f^{(r)}_{q,r-q}$, it follows that

$$(\bar{B}^{(q,r-q)}_{n,m} f)(x, y) \leqslant (B_{n-q,m-r+q} f^{(r)}_{q,r-q})(x, y) + \omega\left(f^{(r)}_{q,r-q}; \frac{q}{n}, \frac{r-q}{m}\right).$$

Subtracting $f^{(r)}_{q,r-q}$ from both sides of the former inequality and by Lemma 1, we obtain

$$|(\bar{B}^{(q,r-q)}_{n,m} f)(x, y) - f^{(r)}_{q,r-q})(x, y)| \leqslant 3\omega(f^{(r)}_{q,r-q}; (n-q)^{-1/2}, (m-r+q)^{-1/2}) + \omega\left(f^{(r)}_{q,r-q}; \frac{q}{n}, \frac{r-q}{m}\right). \tag{3}$$

Now, if we write

$$D(x, y) = (\bar{B}^{(q,r-q)}_{n,m} f)(x, y) - (\bar{B}^{(q,r-q),e}_{n,m} f)(x, y)$$

it is clear that

$$|D(x, y)| \leqslant \sum_{i=0}^{n-q} \sum_{k=0}^{m-r+q} x^i(1-x)^{n-i} y^k(1-y)^{m-k},$$

and (see [1])

$$\|D\|_p = O((nm)^{-1/p}). \tag{4}$$

On the other hand, for $1 \leqslant p \leqslant \infty$,

$$\|f^{(r)}_{q,r-q} - \bar{B}^{(q,r-q),e}_{n,m} f\|_p \leqslant \|f^{(r)}_{q,r-q} - \bar{B}^{(q,r-q)}_{n,m} f\|_p + \|D\|_p. \tag{5}$$

Now (1) follows from (3), (4), and (5).

In the case that $p = \infty$, since $\binom{n}{i} \geqslant n$, $\binom{m}{k} \geqslant m$ $(1 \leqslant i \leqslant n-1,$ $1 \leqslant k \leqslant m-1)$, we have

$$|D(x, y)| \leqslant \sum_{i=1}^{n-1} \sum_{k=1}^{m-1} x^i(1-x)^{n-i} y^k(1-y)^{m-k} \leqslant (nm)^{-1} \tag{6}$$

by the binomial theorem; in this case by means of inequality (5), and with the aid of (3) and (6), (2) follows. The proof is completed.

Therefore, from the continuity of $f^{(r)}_{q,r-q}$ in S, the right-hand side in (1) and (2) approaches zero uniformly in x and y as n, m approach infinity.

3. Main Results

Theorem 1. *Let* $\max(q, r-q) > 1$, $n_1 = \min(n, m)$, *and let* f *be a function such that its partial derivatives of the first* r *orders exist and are continuous in* S.

(a) *If* $1 \leqslant p < \infty$, *then*

$$\|f^{(r)}_{q,r-q} - B^{(q,r-q),e}_{n,m} f\|_p \leqslant A + B + O(n_1^{-\lambda(p)}), \tag{7}$$

where A *and* B *are as in Lemma* 2 *and*

$$\lambda(p) = \begin{cases} 1 & \text{if } p = 1 \\ 2/p & \text{if } 2 \leqslant p < \infty. \end{cases}$$

(b) *If* $p = \infty$ *and the numbers*

$$n!/(n-q)! \qquad m!/(m-r+q)! \qquad \underset{(x,n^{-1})}{\overset{q}{\triangle}} \; \underset{(y,m^{-1})}{\overset{r-q}{\triangle}} f(u_1, u_2)$$

are integers, where u_1 *is either zero or* $(n-q)/n$ *and* u_2 *is either zero or* $(m-r+q)/m$, *then*

$$\|f^{(r)}_{q,r-q} - B^{(q,r-q),e}_{n,m} f\|_\infty \leqslant A + B + O(n_1^{-1}), \tag{8}$$

where A *and* B *are as in Lemma* 2.

Proof. Let us write

$$\frac{(n-q)!}{n!} \frac{(m-r+q)!}{m!} n^q m^{r-q} = 1 + h_{nm}.$$

Now observe that, for $1 \leqslant p \leqslant \infty$,

$$\begin{aligned} &\|f^{(r)}_{q,r-q} - B^{(q,r-q)}_{n,m} f\|_p \\ &\qquad \leqslant \|f^{(r)}_{q,r-q} - \bar{B}^{(q,r-q)}_{n,m} f\|_p + \|h_{nm} B^{(q,r-q)}_{n,m} f\|_p; \end{aligned} \tag{9}$$

moreover, for the second sum we find

$$|h_{nm} B^{(q,r-q)}_{n,m} f| = O(n^{-1}) + O(m^{-1}). \tag{10}$$

Obviously, for $1 \leqslant p \leqslant \infty$,

$$\begin{aligned} \|f^{(r)}_{q,r-q} - B^{(q,r-q),e}_{n,m} f\|_p \leqslant{}& \|f^{(r)}_{q,r-q} - B^{(q,r-q)}_{n,m} f\|_p \\ &+ \|B^{(q,r-q)}_{n,m} f - B^{(q,r-q),e}_{n,m} f\|_p. \end{aligned} \tag{11}$$

Now substituting (9) in (11), and bearing in mind (3), (4), and (10), (7) follows.

For $p=\infty$ the proof proceeds exactly as above but using now inequality (6). This completes the proof.

Remark. For the cases $\max(q, r-q) \leqslant 1$, since

$$(B_{n,m}^{(0,0)}f)(x,y) = (\bar{B}_{n,m}^{(0,0)}f)(x,y),$$
$$(B_{n,m}^{(1,0)}f)(x,y) = (\bar{B}_{n,m}^{(1,0)}f)(x,y),$$
$$(B_{n,m}^{(0,1)}f)(x,y) = (\bar{B}_{n,m}^{(0,1)}f)(x,y),$$
$$(B_{n,m}^{(1,1)}f)(x,y) = (\bar{B}_{n,m}^{(1,1)}f)(x,y),$$

we get by Lemma 2

$$\|f_{q,r-q}^{(r)} - B_{n,m}^{(q,r-q),e}f\|_p \leqslant A + B + O(n_1^{-2/p}) \qquad (1 \leqslant p < \infty),$$

and

$$\|f_{q,r-q}^{(r)} - B_{n,m}^{(q,r-q),e}f\|_p \leqslant A + B + O(n_1^{-2}) \qquad (p = \infty).$$

Now if we consider the Bernstein polynomial in two variables $B_{n-q,m-r+q}f_{q,r-q}^{(r)}$, corresponding to the function $f_{q,r-q}^{(r)}$, we can establish in a similar way the following results:

THEOREM 2. *Let f be a function such that its partial derivatives of the first r orders exist and are continuous in S.*

(a) *If $1 \leqslant p < \infty$, then*

$$\|f_{q,r-q}^{(r)} - B_{n-q,m-r+q}^{e} f_{q,r-q}^{(r)}\|_p \leqslant A + O((nm)^{-1/p});$$

(b) *If $p=\infty$ and the numbers $f_{q,r-q}^{(r)}(u_1, u_2)$ are integers, where u_i is either zero or one, $i = 1, 2$, then*

$$\|f_{q,r-q}^{(r)} - B_{n-q,m-r+q}^{e} f_{q,r-q}^{(r)}\|_\infty \leqslant A + O((nm)^{-1}).$$

In (a) *and* (b) *A is as Lemma* 2.

Remark. The one-dimensional case and $r=0$ were studied in uniform norms by L. V. Kantorovic [6] and in L_p norms by E. Aparicio [1, 2].

ACKNOWLEDGMENT

The author thanks the referee for his helpful suggestions. It was pointed out by the referee that an inequality of the form given in Lemma 1, with a better coefficient, has been given by A. F. Ipatov [10].

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